Torus on-off intermittency in coupled Lorenz systems

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In this paper we describe an example of on-off intermittency linked with the destruction of the torus attractor of a pair of coupled chaotic Lorenz systems. We show that this intermittency has similar properties to on-off intermittency linked with the destruction of the synchronized chaotic attractor located on the system invariant manifold and propose a possible mechanism for it. [S1063-651X(97)09512-3]

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The study of coupled chaotic systems has become very popular among researchers in recent years, stimulated by the various applications [1–10]. As an example, a particular result of some importance is that two identical chaotic systems $\dot{x}=f(x)$ and $\dot{y}=f(y)$, $x,y\in\mathcal{R}^n$, $n\geqslant 3$, evolving on an asymptotically stable chaotic attractor A, when one-to-one coupling

$$\dot{x}=f(x)+d_1(y-x),\quad \dot{y}=f(y)+d_2(x-y) \tag{1}$$

is introduced, can be synchronized for some ranges of $d_{1,2}\in\mathcal{R}$, i.e., $|x(t)-y(t)|\to 0$ as $t\to\infty$ [1].

In the synchronized regime the dynamics of the coupled system (1) is restricted to n-dimensional invariant subspace $x=y$, so the problem of synchronization of chaotic systems can be understood as a problem of stability of an n-dimensional chaotic attractor A in $2n$-dimensional phase space [8–10]. The dynamics of the system (1) is described by two sets of Lyapunov exponents. One of them $\lambda^1=(\lambda_1,\ldots,\lambda_n)$ describes the evolution on the invariant manifold $x=y$ and at least one of the exponents λ_i, $i=1,2,\ldots,n$, is always positive. The second set $\lambda^2=(\lambda_{n+1},\ldots,\lambda_{2n})$ contains exponents that characterize evolution transverse to this manifold and is called transversal. If all transversal Lyapunov exponents are negative, the invariant set A is an attractor, at least in the weak Milnor sense [11].

Shortly after the moment when the largest transversal Lyapunov exponent becomes positive one can observe the phenomenon of on-off intermittency, which is characterized by temporarily intermittent bursting out of the attractor A (invariant manifold $x=y$) and relatively long evolution near A [12–16]. This form of bifurcation leading to the on-off intermittency has been also called blowout bifurcation [15,16]. On-off intermittency has been found to be typical for systems with an invariant manifold (or manifolds) and is one of the ways in which a chaotic attractor located on the invariant manifold can lose its stability.

In this paper we show that a similar mechanism of on-off intermittency can be observed when the attractor of the system (1), which in this case is not an attractor of subsystem $\dot{x}=f(x)$ or $\dot{y}=f(y)$ and so is not located on the invariant manifold, loses stability. We consider the dynamics of two different Lorenz systems coupled by nonsymmetrical one-to-one coupling

$$\begin{aligned}
\frac{dx_1}{dt}&=-\sigma(x_1-y_1)+d_1(x_2-x_1),\\
\frac{dy_1}{dt}&=-x_1z_1+r_1x_1-y_1+d_1(y_2-y_1),\\
\frac{dz_1}{dt}&=x_1y_1-bz_1+d_1(z_2-z_1),\\
\frac{dx_2}{dt}&=-\sigma(x_2-y_2)+d_2(x_1-x_2),\\
\frac{dy_2}{dt}&=-x_2z_2+r_2x_2-y_2+d_2(y-y_2),\\
\frac{dz_2}{dt}&=x_2y_2-bz_2+d_2(z-z_2),
\end{aligned} \tag{2}$$

where $\sigma,b,r_{1,2},d_{1,2}\in\mathcal{R}$ are constant. We assume that each uncoupled system ($d_1=d_2=0$) evolves on chaotic attractors. We have discussed the possibility of chaos synchronization in such a system and its geophysical implications in [17]; in [18] we showed that the attractor of two chaotic systems coupled in this way can be reduced to the fixed point.

In our numerical simulations we took $\sigma=10$, $b=\frac{8}{3}$, $r_1=197$, $r_2=150$, and $d_2=3$ and considered d_1 as a control parameter. For $d_1=0.960$ the evolution of the system takes place on the three-dimensional torus T. The spectrum of Lyapunov exponents of the typical orbit can be divided into two subsets: λ^1 ($\lambda_1=0$, $\lambda_2=0$, and $\lambda_3=-0.1$) describing evolution on the torus and λ^2 ($\lambda_4=-7.42$, $\lambda_5=-14.34$, and $\lambda_6=-17.36$) describing evolution transverse to the torus. A two-dimensional cross section of the torus T is shown in Fig. 1.

At $d_1^c=1.019$ the torus breaks down and we observe on-off intermittency in the d_1 interval [1.019,1.100]. During this intermittency a typical trajectory evolves for some relatively long period of time τ_T in the neighborhood of the broken torus and occasionally bursts out of it. A two-dimensional

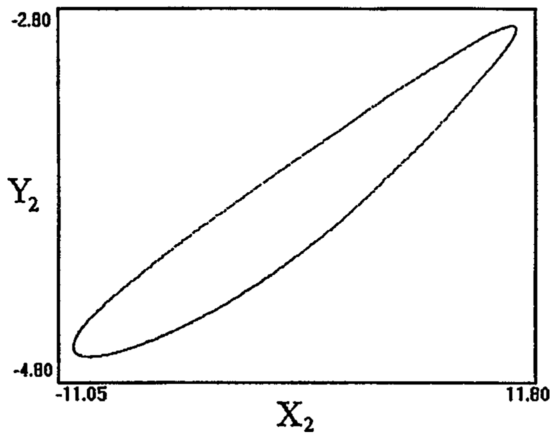

FIG. 1. Two-dimensional $x_1 - x_2$ cross section of the torus T. $\sigma = 10$, $b = \frac{8}{3}$, $r_1 = 197$, $r_2 = 150$, $d_1 = 0.960$, and $d_2 = 3$.

cross section of the neighborhood of the destroyed torus T and a time series typical for this behavior are shown in Figs. 2(a) and 2(b). In this case the spectrum of Lyapunov exponents is as follows: $\lambda_1 = 0.29$, $\lambda_2 = 0$, $\lambda_3 = -0.49$, $\lambda_4 = -7.29$, $\lambda_5 = -14.19$, and $\lambda_6 = -17.67$ and the system trajectory evolves on the chaotic attractor located in the neighborhood of the destroyed torus T.

At the beginning of the intermittency at $d_1^c = 1.019$, one of the transverse Lyapunov exponents becomes positive, so a nonzero measure set of points with one unstable direction occurs on the torus. The system trajectory entering this set leaves the torus along the unstable manifold and after an evolution out of the neighborhood of the destroyed torus T (the burst) is diverted back into this neighborhood. The average time period of the evolution in the neighborhood of the broken torus τ_T scales with the distance from the bifurcation point $(d_1 - d_1^c)$ as

$$\tau_T \sim (d_1 - d_1^c)^{1/2}. \quad (3)$$

This relation is the same as for classical intermittency of type I (connected with saddle-node bifurcation).

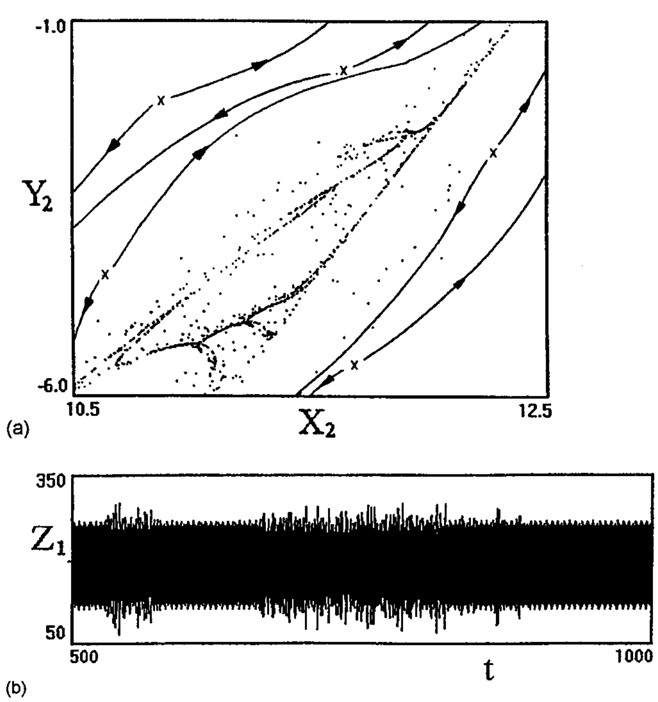

FIG. 2. (a) Two-dimensional $x_1 - x_2$ cross section of the torus T. $\sigma = 10$, $b = \frac{8}{3}$, $r_1 = 197$, $r_2 = 150$, $d_1 = 1.020$, and $d_2 = 3$. $\times$ indicates unstable periodic orbits of saddle type. (b) Time series of torus on-off intermittency.

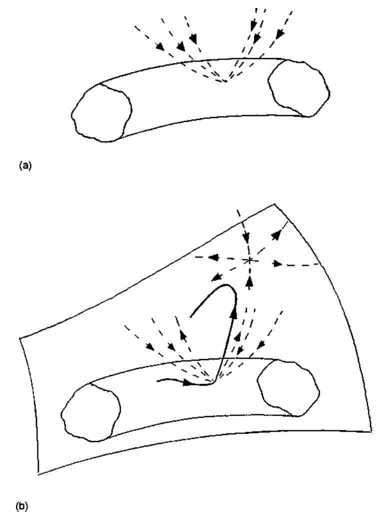

FIG. 3. Mechanism of torus on-off intermittency (a) before bifurcation and (b) after bifurcation.

The mechanism of the observed on-off intermittency based on torus T can be explained in the following way. During a torus breakdown bifurcation a positive measure set $\mathcal{A}$ of points on the torus undergoes a bifurcation schematically described in Figs. 3(a) and 3(b), after which one of the directions transverse to the torus becomes unstable. Simultaneously, in the neighborhood of the torus, unstable periodic orbits of a saddle type (with at least one stable direction) are formed. The envelope of unstable manifolds of these orbits forms the threshold for the evolution of the trajectory, makes the expansion too far from the neighborhood of the destroyed torus impossible, and enforces the trajectory's return to this neighborhood.

The mechanism described is the generalization to higher-dimensional phase space of the bubbling transition model, which we have recently identified in coupled logistic maps [19]. In this work we prove that the transition point to on-off intermittency after the destruction of a synchronized chaotic attractor is associated with the occurrence of an unstable periodic orbit of saddle type; its unstable manifold bounds the region near the destroyed attractor, so that the system trajectory is not allowed to leave it.

In the present system, due to the six-dimension phase space, it is impossible to prove that the proposed mechanism is the only possible way to on-off intermittency. We have found some numerical evidence that our mechanism is possible by identifying several unstable orbits in the neighborhood of the destroyed torus and estimating their unstable manifolds on the two-dimensional cross section. These results are shown in Fig. 2(a). The conjectured envelope of the unstable manifolds surrounds the region of the phase space where the destroyed torus was located. At the end of the on-off intermittency interval some further bifurcation on the unstable periodic orbits of saddle type can lead to the creation of holes in the envelope or allow creation of stable attractors in different regions of the phase space.

In summary, we have presented a model consisting of two coupled chaotic Lorenz systems that exhibits on-off intermittency after the destruction of a torus attractor that is not located on the invariant manifold. (Our example shows that on-off intermittency can occur also after the destruction of regular attractors that are not located on the invariant manifold.) Near the transition point, shortly after one of the transverse Lyapunov exponents becomes positive, we observed universal scaling behavior with a scaling exponent equal to the characteristic for classical type-I intermittency. We also proposed a mechanism based on the bifurcation of saddle periodic orbits that can be responsible for the observed intermittency.

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